GALOIS EMBEDDING OF K3 SURFACE
– ABELIAN CASE –

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Abstract. We study Glois embeddings of $K3$ surfaces in the case where the Galois groups are abelian. We show several properties of $K3$ surfaces concerning the Galois embeddings. In particular, if the Galois group G is abelian, then $G \cong \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ and S is a smooth complete intersection of hypersurfaces. Further, we state the detailed structure of such surfaces.

1. Introduction

The purpose of this article is to study Galois embeddings of $K3$ surfaces, where the Galois groups are abelian. The non-abelian case will be treated later. Before going into the study on $K3$ surfaces, we recall the definition of Galois embeddings of algebraic varieties and their properties.

Let k be the ground field of our discussion, we assume it to be the field of complex numbers, however most results hold also for an algebraically closed field of characteristic zero. Let V be a nonsingular projective algebraic variety of dimension n with a very ample divisor D, we denote this by a pair (V, D). Let $f = f_D : V \hookrightarrow \mathbb{P}^N$ be the embedding of V associated with the complete linear system $|D|$, where $N + 1 = \dim \mathrm{H}^0(V, \ \mathcal{O}(D))$. Suppose that W is a linear subvariety of $\mathbb{P}^N$ satisfying $\dim W = N - n - 1$ and $W \cap f(V) = \emptyset$. Consider the projection π_W from W to $\mathbb{P}^n$, $\pi_W : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$. Restricting π_W onto $f(V)$, we get a surjective morphism $\pi = \pi_W \cdot f : V \longrightarrow \mathbb{P}^n$.

Let $K = k(V)$ and $K_0 = k(\mathbb{P}^n)$ be the function fields of V and $\mathbb{P}^n$ respectively. The morphism π induces a finite extension of fields $\pi^* : K_0 \hookrightarrow K$ of degree $d = \deg f(V) = D^n$, which is the self-intersection number of D. We denote by K_W the Galois closure of this extension and by $G_W = Gal(K_W/K_0)$ the Galois group of K_W/K_0. By [1] we see that G_W is isomorphic to the monodromy group of the covering $\pi : V \longrightarrow \mathbb{P}^n$. Let V_W be the K_W-normalization of V (cf. [2, Ch.2]). Note that V_W is determined uniquely by V and W.

Definition 1.1. In the above situation we call G_W and V_W the Galois group and the Galois closure variety at W respectively (cf. [12]). If the extension K/K_0 is Galois, then we call f and W a Galois embedding and a Galois subspace for the embedding respectively.

Definition 1.2. A nonsingular projective algebraic variety V is said to have a Galois embedding if there exist a very ample divisor D satisfying that the embedding associated with $|D|$ has a Galois subspace. In this case the pair (V, D) is said to define a Galois embedding.

If W is the Galois subspace and T is a projective transformation of $\mathbb{P}^N$, then $T(W)$ is a Galois subspace of the embedding $T \cdot f$. Therefore the existence of Galois subspace does not depend on the choice of the basis giving the embedding.

Remark 1.3. If a smooth variety V exists in a projective space, then by taking a linear subvariety, we can define a Galois subspace and Galois group similarly as above. Suppose that V is not normally embedded and there exists a linear subvariety W such that the projection π_W induces a Galois extension of fields . Then, taking D as a hyperplane section of V in the embedding, we infer readily that (V, D) defines a Galois embedding with the same Galois group in the above sense.

By this remark, for the study of Galois subspaces, it is sufficient to consider the case where V is normally embedded.

We have studied Galois subspaces and Galois groups for hypersurfaces in [8], [9] and [10] and space curves in [11] and [13]. The method introduced in [12] is a generalization of the ones in these studies.

Hereafter we use the following notation and convention:
· $\mathrm{Aut}(V)$: the automorphism group of a variety V
· $|G|$: the order of a group G
· $\sim$: the linear equivalence of divisors
· $\mathbf{1}_m$: the unit matrix of size m
· $[\alpha_1, \ldots, \alpha_m]$: the diagonal matrix with entries $\alpha_1, \ldots, \alpha_m$

The organization of this article is as follows: In Section 2 we review the results of Galois embeddings, which will be used in the sequel. We devote the remainder sections to the study of the Galois embedding of $K3$ surfaces.

2. Results on Galois embeddings

We state several properties concerning Galois embedding without proofs, for the details see [12]. By definition, if W is a Galois subspace, then each element σ of G_W is an automorphism of $K = K_W$ over K_0. Therefore it induces a birational transformation of V over $\mathbb{P}^n$. This implies that G_W can be viewed as a subgroup of $\mathrm{Bir}(V/\mathbb{P}^n)$, the group of birational transformations of V over $\mathbb{P}^n$. Further we can say the following:

Representation 1. *Each birational transformation belonging to G_W turns out to be regular on V, hence we have a faithful representation*

$$\alpha : G_W \hookrightarrow \mathrm{Aut}(V). \tag{1}$$

Therefore, if the order of $\mathrm{Aut}(V)$ is smaller than the degree d, then (V, D) cannot define a Galois embedding. In particular, if $\mathrm{Aut}(V)$ is trivial, then V has no Galois embedding. On the other hand, in case V has infinitely many automorphisms, we

have examples such that there exist infinitely many distinct Galois embeddings, see Example 4.1 in [12].

When (V, D) defines a Galois embedding, we often identify $f(V)$ with V. Let H be a hyperplane of $\mathbb{P}^N$ containing W. Let D' be the intersection divisor of V and H. Since $D' \sim D$ and $\sigma^*(D') = D'$, for any $\sigma \in G_W$, we see that σ induces an automorphism of $\mathrm{H}^0(V, \mathcal{O}(D))$. This implies the following.

Representation 2. *We have a second faithful representation*

$$\beta : G_W \hookrightarrow PGL(N, \mathbb{C}). \tag{2}$$

In the case where W is a Galois subspace we identify $\sigma \in G_W$ with $\beta(\sigma) \in PGL(N, \mathbb{C})$ hereafter. Since G_W is a finite subgroup of $\mathrm{Aut}(V)$, we can consider the quotient V/G_W and let π_G be the quotient morphism, $\pi_G : V \longrightarrow V/G_W$.

Proposition 2.1. *If (V, D) defines a Galois embedding with the Galois subspace W such that the projection is $\pi_W : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$, then there exists an isomorphism $g : V/G_W \longrightarrow \mathbb{P}^n$ satisfying $g \cdot \pi_G = \pi$. Hence the projection π turns out to be a finite morphism and the fixed loci of G_W consist of only divisors.*

Therefore, π is a Galois covering in the sense of Namba [6]. We have a criterion that (V, D) defines a Galois embedding.

Theorem 2.2. *The pair (V, D) defines a Galois embedding if and only if the following conditions hold*:

1. *There exists a subgroup G of $\mathrm{Aut}(V)$ satisfying that $|G| = D^n$.*
2. *There exists a G-invariant linear subspace $\mathcal{L}$ of $\mathrm{H}^0(V, \mathcal{O}(D))$ of dimension $n+1$ such that, for any $\sigma \in G$, the restriction $\sigma^*|_{\mathcal{L}}$ is a multiple of the identity.*
3. *The linear system $\mathcal{L}$ has no base points.*

It is easy to see that $\sigma \in G_W$ induces an automorphism of W, hence we obtain another representation of G_W as follows. Take a basis $\{f_0, f_1, \ldots, f_N\}$ of $\mathrm{H}^0(V, \mathcal{O}(D))$ satisfying that $\{f_0, f_1, \ldots, f_n\}$ is a basis of $\mathcal{L}$ in Theorem 2.2. Then we have the representation

$$\beta_1(\sigma) = \begin{pmatrix} \lambda_\sigma & & & \vdots & \\ & \ddots & & \vdots & * \\ & & \lambda_\sigma & \vdots & \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ & \mathbf{0} & & \vdots & M'_\sigma \end{pmatrix}. \tag{3}$$

Since the representation is completely reducible, we get another representation using a direct sum decomposition:

$$\beta_2(\sigma) = \lambda_\sigma \cdot \mathbf{1}_{n+1} \oplus M'_\sigma.$$

Thus we can define

$$\gamma(\sigma) = M'_\sigma \in PGL(N - n - 1, \mathbb{C}).$$

Therefore σ induces an automorphism on W given by M'_σ.

Representation 3. *We get a third representation*

$$\gamma : G_W \longrightarrow PGL(N-n-1, \mathbb{C}). \tag{4}$$

Let G_1 and G_2 be the kernel and image of γ respectively.

Theorem 2.3. *We have an exact sequence of groups*

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{\gamma} G_2 \longrightarrow 1,$$

where G_1 is a cyclic group.

Corollary 2.4. *If $N = n+1$, i.e., $f(V)$ is a hypersurface, then G is a cyclic group.*

This assertion has been obtained in [10]. Moreover we have another representation.

Suppose that (V, D) defines a Galois embedding and let G be a Galois group for some Galois subspace W. Then, take a general hyperplane W_1 of $\mathbb{P}^n$ and put $V_1 = \pi^*(W_1)$. The divisor V_1 has the following properties:

(i) If $n \geq 2$, then V_1 is a smooth irreducible variety.
(ii) $V_1 \sim D$.
(iii) $\sigma^*(V_1) = V_1$ for any $\sigma \in G$.
(iv) V_1/G is isomorphic to W_1.

Put $D_1 = V_1 \cap H_1$, where H_1 is a general hyperplane of $\mathbb{P}^N$. Then (V_1, D_1) defines a Galois embedding with the Galois group G (cf. Remark 1.3). Iterating the above procedures, we get a sequence of pairs (V_i, D_i) such that

$$(V, D) \supset (V_1, D_1) \supset \cdots \supset (V_{n-1}, D_{n-1}).$$

These pairs satisfy the following properties:

(a) V_i is a smooth subvariety of V_{i-1}, which is a hyperplane section of V_{i-1}, where $D_i = V_{i+1}$, $V = V_0$ and $D = V_1$ $(1 \leq i \leq n-1)$.
(b) (V_i, D_i) defines a Galois embedding with the same Galois group G.

In particular, letting C be the curve V_{n-1}, we get the following fourth representation.

Representation 4. *We have a fourth faithful representation*

$$\delta : G_W \hookrightarrow \mathrm{Aut}(C), \tag{5}$$

where C is a smooth curve in V given by $V \cap L$ such that L is a general linear subvariety of $\mathbb{P}^N$ with dimension $N-n+1$ containing W.

Note that in some cases there exist several Galois subspaces and Galois groups for one embedding (see, for example [13]). Generally we have the following.

Proposition 2.5. *Suppose that (V, D) defines a Galois embedding and let W_i $(i = 1, 2)$ be Galois subspaces such that $W_1 \neq W_2$. Then $G_1 \neq G_2$ in* $\mathrm{Aut}(V)$*, where G_i is the Galois group at W_i.*

Corollary 2.6. *If V is a smooth projective algebraic variety of general type, then there are at most finitely many Galois subspaces.*

Remark 2.7. It may happen that there exist infinitely many Galois subspaces for one embedding if the Kodaira dimension of V is small. For example, if $V = \mathbb{P}^1$ and $\deg D = 3$, i.e., $f(V)$ is a twisted cubic, then the Galois lines form two dimensional locally closed subvariety of the Grassmannian $\mathbb{G}(1,3)$, parametrizing lines in projective three space (cf. [11]).

3. $K3$ Surfaces

We apply the methods developed in the previous sections to the study of $K3$ surfaces. For each abelian surface with a Galois embedding, we have studied in detail in [12]. In particular, we have given the complete list of the complex representation of every possible group and shown that the surface is isogenous to the square of an elliptic curve.

A curve and surface will mean a nonsingular projective algebraic curve and surface respectively. In addition to the notation listed in Section 1, we use the following hereafter:

· $\langle a_1, \cdots, a_m \rangle$: the subgroup generated by $a_1, \cdots, a_m$
· Z_m : the cyclic group of order m
· $e_m := \exp(2\pi\sqrt{-1}/m)$
· $D_1.D_2$: the intersection number of two divisors D_1 and D_2 on a surface
· D^2 : the self-intersection number of a divisor D on a surface
· $(X_0 : \cdots : X_m)$: a set of homogeneous coordinates on $\mathbb{P}^m$
· $g(C)$: the genus of a smooth curve C
· Supp D : the support of a divisor D
· $X_{(4)}$: a smooth quartic surface in $\mathbb{P}^3$
· $X_{(23)}$: a smooth $(2,3)$-complete intersection of hypersurfaces in $\mathbb{P}^4$
· $X_{(222)}$: a smooth $(2,2,2)$-complete intersection of hypersurfaces in $\mathbb{P}^5$

Suppose that S is a $K3$ surface such that (S, D) defines a Galois embedding with the Galois group $G \subset \mathrm{Aut}(S)$. Let ω_S be a nowhere vanishing holomorphic two form of S Then, let $\varepsilon : G \longrightarrow \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the character of the natural representation of G on the space $H^{2,0}(S) = \mathbb{C}\omega_S$, i.e., $\varepsilon(\sigma) = \lambda$ for $\sigma \in G$ if $\sigma^*(\omega_S) = \lambda\omega_S$. There exists a multiplicative group Γ_m of the m-th roots of unity and the following exact sequence of groups:

$$1 \longrightarrow G_s \longrightarrow G \xrightarrow{\varepsilon} \Gamma_m \longrightarrow 1, \tag{6}$$

where G_s is a symplectic group [5]. Let $\pi : S \longrightarrow \mathbb{P}^2$ be the projection, which is a Galois covering defined in Section 2. Let W be the center of the projection and H a general hyperplane containing W. Put $C = S \cap H$. Then C is an irreducible smooth curve and $C \sim D$.

Lemma 3.1. *The representation $r : G \longrightarrow \mathrm{Aut}(C)$ given by $r(\sigma) = \sigma|_C$ is injective.*

Proof. Note that $\sigma \in G$ is an automorphism of S over $\mathbb{P}^2$ and $\sigma(C) = C$. If $\sigma|_C$ is identity, then C is a component of the ramification divisor of the covering. Since C is given by H which is general, σ must be identity. □

The restriction $\pi|_C : C \longrightarrow \mathbb{P}^1$ turns out to be a Galois covering, where the Galois group is isomorphic to G. Since $\mathrm{H}^1(S, \mathcal{O}) = 0$ and the canonical divisor on S is

trivial, the restriction of f_D to C gives the canonical embedding of C. Therefore C has a Galois embedding given by its canonical divisor.

Lemma 3.2. *The group G is non-symplectic, i.e., $\Gamma_m \neq \{1\}$.*

Proof. Suppose $\Gamma_m = \{1\}$. Then, $G = G_s$. This means that the fixed loci of each element of G is at most finitely many points. This contradicts to Proposition 2.1. $\square$

Let R be the ramification divisor for π.

Lemma 3.3. *We have $R \sim 3D$ and Supp R is connected. Each irreducible component of Supp R is smooth.*

Proof. Since the canonical divisor on S is trivial, using the adjunction formula, we get $\pi^*(-3\ell) + R \sim 0$ for a line ℓ in $\mathbb{P}^2$. Since $\pi^*(\ell) \sim D$, we have $R \sim 3D$, hence R is very ample. $\square$

Example 3.4. Let S be the Fermat quartic surface: $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$ and P be one of the points $(1:0:0:0), (0:1:0:0), (0:0:1:0)$ and $(0:0:0:1)$. The projection from P to the hyperplane $\mathbb{P}^2$ defines a cyclic Galois covering (such P is called a Galois point [10]). Note that S is a Kummer surface $Km(E \times E)$, where $E = \mathbb{C}/(1, e_4)$. Further, it is a singular $K3$ surface, i.e., $\rho(S) = 20$ (cf. [3]).

4. ABELIAN CASE

In the case of Galois embeddings of abelian surfaces, the group cannot be abelian. However, in the case of $K3$ surfaces, the group can be a cyclic group as in Example 3.4. Nikulin [7] shows that there exist many abelian automorphism groups for $K3$ surfaces. So let us consider the Galois embedding where the Galois group G is abelian. Hereafter we assume G is abelian if not otherwise mentioned.

Theorem 4.1. *If the Galois group G is abelian, then $G \cong Z_4$, Z_6 or ${Z_2}^3 = Z_2 \times Z_2 \times Z_2$ and S is isomorphic to $S_{(4)}$, $S_{(23)}$ or $S_{(222)}$ respectively.*

We will give concrete examples for the three surfaces in Section 5. We note that for the proof of Theorem 4.1 we do not use the property of Galois embedding, but only use that the covering $\pi : S \longrightarrow \mathbb{P}^2$ is Galois (except in the proof of Claim 4.16). So the result may be known, but for the sake of completeness, we give the proof in this article.

Before proceeding with the proof, we fix the notation. Let $\pi : S \longrightarrow \mathbb{P}^2$ be the Galois covering induced by the projection. Put $|G| = n$ and assume that $R = (n_1 - 1)C_1 + \cdots + (n_r - 1)C_r$, where C_i are irreducible components. For $\sigma \in G$ put $F(\sigma) = \{ \, x \in S \mid \sigma(x) = x \, \}$.

Lemma 4.2. *For each point $x \in$ Supp R the stabilizer of the point $G_x = \{ \, \sigma \in G \mid \sigma(x) = x \, \}$ is generated by at most two elements.*

Proof. There exists an open neighbourhood U_x and coordinates on it such that G_x has a representation in $GL(2, \mathbb{C})$. Since G is abelian, we can assume each element of G_x is generated by one or two diagonal matrices $[\alpha, 1]$ and $[1, \beta]$, where $\alpha^n = \beta^n = 1$. $\square$

Lemma 4.3. *The following assertions hold true.*

(1) *Supp R is connected.*
(2) *Each irreducible component is a smooth curve.*
(3) *Supp R has normal crossings.*

Proof. Since $R \sim 3D$, we have R is ample, hence Supp R is connected. Each component C_i is given by $\{\ x \in S \mid \sigma(x) = x\ \}$ for some $\sigma \in G \setminus \{id\}$, where locally σ can be expressed as a diagonal matrix $[\alpha, 1]$. Thus C_i is smooth. Suppose $x \in$Supp R is an intersection point of some components C_i $(1 \le i \le r)$. As we have seen in the proof of Lemma 4.2, there exist two elements $[\alpha, 1]$ and $[1, \beta]$ which are generators of the stabilizer G_x. Thus there exist just two irreducible components meeting normally. □

Lemma 4.4. *For each irreducible component C of R, we have Supp $\pi^*(\pi(C)) = C$, i.e., $\tau(C) = C$ for any $\tau \in G$. In particular C is an ample divisor.*

Proof. Let $\sigma \in G$ satisfy $\sigma \neq id$ and $\sigma|_C = id$. Since $\pi(C)$ is ample and $\pi : S \longrightarrow \mathbb{P}^2$ is a finite morphism, $\pi^*(\pi(C))$ is ample and hence Supp $\pi^*(\pi(C))$ is connected. Suppose Supp $\pi^*(\pi(C))$ is reducible. Then, there exists another irreducible component C' of $\pi^*(\pi(C))$ such that $C' = \sigma'(C)$ for some $\sigma' \in G$ and $C \cap C' \neq \emptyset$. Since $\sigma\sigma' = \sigma'\sigma$, we have $\sigma(\sigma'(y)) = \sigma'(y)$ for any $y \in C$. This means $\sigma|_{C'} = id$. Take $x \in C \cap C'$. Then C and C' have a normal crossing at x by Lemma 4.3. However, looking at σ near x, the σ can be expressed as one of the diagonal matrices $[\alpha, 1]$ and $[1, \beta]$, where $\alpha \neq 1$ and $\beta \neq 1$. This contradicts to that $\sigma|_{C'} = id$. □

Corollary 4.5. *With the same notation as in Lemma 4.4, we have $C^2 > 0$, hence $g(C) \ge 2$.*

Proof. Since $\pi^*(\pi(C))$ can be expressed as mC, we have $m^2C^2 = n(\pi(C))^2 \ge n$. Since $2g(C) - 2 = C^2$ we have the assertion. □

Put $G_i = \{\ \sigma \in G \mid \sigma|_{C_i} = id\ \}$ $(1 \le i \le r)$. Then G_i is determined uniquely by C_i and not a trivial subgroup of G.

Lemma 4.6. *The group G_i is cyclic.*

Proof. For a general point $x \in C_i$, taking a suitable local coordinates, we can express each $\sigma \in G_i$ as $[\alpha, 1]$. We have a monomorphism $\rho : G_i \longrightarrow \mathbb{C}^\times$, where $\rho(\sigma) = \alpha$. Since $\rho(G_i)$ is a cyclic group, so is G_i. □

Lemma 4.7. *The surface $S_i = S/G_i$ $(1 \le i \le r)$ is a smooth rational surface.*

Proof. Since near each point $x \in C_i$, the $\sigma \in G_i$ can be expressed as a diagonal matrix. Hence S_i is smooth. Let K_i be a canonical divisor on S_i. Then, we have $\pi_i^*(K_i) + R_i \sim 0$, where $\pi_i : S \longrightarrow S_i$ and R_i is a ramification divisor for π_i. Since R_i is effective, we infer that $\dim \mathrm{H}^0(S_i, \mathcal{O}(2K_i)) = 0$. Clearly we have $\dim \mathrm{H}^0(S_i, \Omega_i^1) = 0$, where Ω_i^1 is the sheaf of holomorphic 1-forms on S_i. Therefore S_i is rational by Castelnuovo's Rationality Criterion. □

Lemma 4.8. *There does not exist $\tau \in G$ such that $\tau \neq id$ and $F(\tau) = \emptyset$.*

Proof. Suppose otherwise. Since G is abelian, expressing $G \cong \langle\tau\rangle \times G'$, we put $S' = S/G'$. Then S' is a smooth rational surface. Because, as we see in the proof of Lemma 4.2, G_x is generated locally by reflections. Hence S' is smooth. Since there exists a covering $S_i \longrightarrow S'$ (or $S_i = S'$) and S_i is rational, we see that S'

is rational. The $\pi' : S' \longrightarrow \mathbb{P}^2 = S/G'$ is an unramified double covering, this is a contradiction. $\square$

Lemma 4.9. *The group G_i $(1 \le i \le r)$ determines C_i uniquely and $G_i \cap G_j$ consists of identity if $i \neq j$. Therefore, there exists a one to one correspondence between the set $\{\ G_i \mid 1 \le i \le r\}$ and $\{\ C_i \mid 1 \le i \le r\ \}$.*

Proof. If $C_i \neq C_j$, then we have $C_i \cap C_j \neq \emptyset$ by Lemma 4.4. Take $x \in C_i \cap C_l$. Consider G_i and G_j in a neighbourhood of x. Since G is abelian, there exist generators $[\alpha, 1]$ and $[1, \beta]$ of G_i and G_j respectively. If $\sigma \in G_i \cap G_j$, then $\sigma|_{C_i} = \sigma|_{C_j} = id$. This implies that $\sigma = id$. $\square$

Lemma 4.10. *The group G can be expressed as a direct product $G_1 \times \cdots \times G_r$, where each G_i is cyclic $(1 \le i \le r)$.*

Proof. For each element $\sigma \in G$, there exists a fixed point of σ by Lemma 4.8. If $F(\sigma)$ contains a curve, then there exist i such that $\sigma|_{C_i} = id$. This means that $\sigma \in G_i$. On the other hand, if $F(\sigma)$ consists of only points, then take $x \in F(\sigma)$. It is easy to see that there exist two curves C_i and C_j containing x. Then σ can be expressed as a product of elements of G_i and G_j. Therefore, we conclude the assertion from Lemma 4.9. $\square$

Let Δ_i be the plane curve $\pi(C_i)$ and put $\Delta = \Delta_1 + \cdots + \Delta_r$.

Lemma 4.11. *Each Δ_i is smooth $(1 \le i \le r)$ and Δ has normal crossings.*

Proof. In the proof of Lemma 4.4, we have shown that $\tau(C_i) = C_i$ for each $\tau \in G$. Therefore G acts on C_i and we can consider C_i/G. We denote it by Δ_i. Hence Δ_i is smooth. For a point $x \in C_i \cap C_j$ we have $\sigma_i(x) = \sigma_j(x) = x$ and $\sigma_k(C_i) = C_i$ and $\sigma_k(C_j) = C_j$ $(1 \le i, j, k \le r)$. Hence Δ has normal crossings. $\square$

Put $n_i = |G_i|$. Then we have $n = \prod_{i=1}^r n_i$ by Lemma 4.10. Denote by $\chi(V)$ the topological Euler characteristic of a curve or a surface V.

Lemma 4.12. *We have $n_i = 2, 3$ or 4 for each i.*

Proof. Put $\bar{C}_i = \pi_i(C_i)$ where $\pi_i : S \longrightarrow S_i = S/G_i$. Compare $\chi(S)$ and $\chi(S_i)$. Since $G_i = \langle \sigma_i \rangle$ and $\sigma_i|_{C_i} = id$, the C_i is isomorphic to $\bar{C}_i$. Hence we have

$$\begin{aligned} \chi(S) &= \chi(S - C_i) + \chi(C_i) \\ &= n_i \chi(S_i - \bar{C}_i) + \chi(\bar{C}_i) \\ &= n_i \chi(S_i) + (1 - n_i)\chi(\bar{C}_i) \end{aligned}$$

We have $\chi(C_i) = 2 - 2g(C_i) = \chi(\bar{C}_i)$. Therefore we have

$$24 = n_i \chi(S_i) + (n_i - 1)(2g(C_i) - 2). \tag{7}$$

Since S_i is a smooth rational surface by Lemma 4.7, we have $\chi(S_i) \ge 3$. Further, we have $g(C_i) \ge 2$ by Corollary 4.5. Thus, clearly we have $n_i \le 5$. In case $n_i = 5$, we have $24 = 5\chi(S_i) + 8(g(C_i) - 1)$, but this cannot hold. Whence we conclude $n_i \le 4$. $\square$

Next we consider the branch divisor for π. Put $d_i = \deg \Delta_i$ $(1 \le i \le r)$.

Lemma 4.13. *We have the equality*

$$d_1\left(1-\frac{1}{n_1}\right)+\cdots+d_r\left(1-\frac{1}{n_r}\right)=3. \tag{8}$$

In particular, we have $r \le 6$.

Proof. Letting ℓ be a line in $\mathbb{P}^2$, we have $\pi^*(3\ell).\pi^*(\ell)=3n$. Since $\pi^*(\Delta_i)=n_iC_i$, we have $n_iC_i.\Gamma=\pi^*(\Delta_i).\Gamma=nd_i$, where $\Gamma=\pi^*(\ell)$. Since $R\sim 3D$ by Lemma 3.3 and $D\sim\pi^*(\ell)$, we get

$$(n_1-1)\Gamma C_1+\cdots+(n_r-1)\Gamma C_r=3n,$$

hence

$$(n_1-1)\frac{d_1}{n_1}n+\cdots+(n_r-1)\frac{d_r}{n_r}=3n.$$

This proves the equation. Since $n_i\ge 2$ and $d_i\ge 1$, we have $r\le 6$. $\square$

Lemma 4.14. *We have $d_i \ge 2$ for each i.*

Proof. Put $\hat{G}_i=G/G_i$ and consider the coverings

$$p_i: S\longrightarrow S/\hat{G}_i=\hat{S}_i \ \text{ and } \ q_i:\hat{S}_i\longrightarrow\mathbb{P}^2=S/G.$$

By Lemma 4.4 $\hat{G}_i$ acts on C_i, hence put $\hat{C}_i=p_i(C_i)=C_i/\hat{G}_i$. By repeating the similar arguments as in the proof of Lemma 4.7, we conclude $\hat{S}_i$ is a smooth rational surface. Suppose $d_i=1$, Then, $q_i(\hat{C}_i)=\Delta_i$ is a line ℓ. Hence we get $q_i^*(\ell)=n_i\hat{C}_i$. This means $q_i^*(\ell)^2=n_i=n_i^2\hat{C}_i{}^2$. Hence $n_i\hat{C}_i{}^2=1$, i.e., $n_i=1$. This is a contradiction. $\square$

Making use of Lemmas 4.12, 4.13 and 4.14, we determine r, n_i and d_i $(1\le i\le r)$.

Claim 4.15. *If there exists i such that $n_i=4$, then $G\cong Z_4$.*

Proof. We may assume $i=1$. We prove the claim by examining the cases:

(i) $r=1$.
(ii) There exists $j\ge 2$ such that $n_j=4$.
(iii) $n_j\le 3$ for all $j\ge 2$ and there exists for some $j\ge 2$ such that $n_j=3$
(iv) $n_j=2$ for all $j\ge 2$

In case of (i) we observe the equality (7). We have $24=4\chi(S_1)+6(g(C_1)-1)$. Since $g_1=g(C_1)\ge 2$, we have $g_1=3$ and $\chi(S_1)=3$. Since S_1 is a smooth rational surface, we have $S_1\cong\mathbb{P}^2$, hence $d_1=4$. Therefore we have $G\cong Z_4$.

The case (ii) does not occur. Suppose otherwise. Then, since $d_j\ge 2$, we infer from (8) that $r=2$, $d_1=d_2=2$ and $n_1=n_2=4$. Note that $\sigma_1|_{C_1}=id$ and σ_1 acts on C_2. Thus we have $\chi(\Delta_i)=2$ and $\chi(C_i)=-4$ $(i=1,2)$. Then we get

$$\begin{aligned}\chi(S) &= \chi(S-(C_1\cup C_2))+\chi(C_1\cup C_2)\\ &= 16\chi(\mathbb{P}^2-(\Delta_1\cup\Delta_2))+\chi(\Delta_1)+\chi(\Delta_2)-\chi(\Delta_1\cap\Delta_2)\\ &= 36.\end{aligned}$$

which is a contradiction.

The case (iii) does not occur. Suppose otherwise. Then, from (8) we have

$$3\ge d_1\left(1-\frac{1}{4}\right)+d_2\left(1-\frac{1}{3}\right)\ge\frac{3}{2}+\frac{2}{3}d_j.$$

Since $d_j \neq 1$, we have $d_j = 2$. This means that $r = 2$ and $3d_2/4 = 5/3$, which is a contradiction.

The case (iv) does not occur. Suppose otherwise. Then, from (8) we have that $3d_1 + 2(d_2 + \cdots + d_r) = 12$, which implies that $r = 2$ and $d_2 = 3$, i.e., $n_1 = 4, d_1 = 2$ and $n_2 = 2, d_2 = 3$. Note that $\sigma_1|_{C_1} = id$ and σ_1 acts on C_2. We infer readily that $C_2' := C_2/G$ is a smooth curve in $S_1 := S/G_1 \cong \mathbb{P}^2$. We have $\pi = q_1 \cdot p_1$, where $p_1 : S \longrightarrow S_1$ and $q_1 : S_1 \longrightarrow S/G \cong \mathbb{P}^2$. Then $q_1 : S_1 \longrightarrow \mathbb{P}^2$ is a double covering branched along just Δ_2, which is cubic. This is a contradiction. □

Therefore we assume $n_i = 2$ or 3. Since $d_i \geq 2$, we have $r \leq 3$.

(1) In case $r = 3$, it is easy to see that $d_i = n_i = 2$ for $i = 1, 2, 3$. Then, $G \cong {Z_2}^3$.

(2) In case $r = 2$, we have

$$d_1\left(1 - \frac{1}{n_1}\right) + d_2\left(1 - \frac{1}{n_2}\right) = 3.$$

Then we have $5 \leq d_1 + d_2 \leq 6$. We assume $d_1 \geq d_2$ and find the solutions. Here we use the notation $(a, b; c, d)$, which means $a = d_1, b = n_1$ and $c = d_2, d = n_2$.

(b-1) In the case $d_1 + d_2 = 6$, we have $(4, 2; 2, 2)$ or $(3, 2; 3, 2)$.

(b-2) In the case $d_1 + d_2 = 5$, we have $(3, 3; 2, 2)$ or $(3, 2; 2, 4)$.

Claim 4.16. *The case $r = 2$ and $n_1 = n_2 = 2$ does not occur.*

Proof. We show that G cannot be isomorphic to $Z_2 \times Z_2$. Suppose (S, D) gives a Galois embedding. Then, $|G| = D^2 = 4$ and $\dim \mathrm{H}^0(S,\ \mathcal{O}(D) = 4$. Thus $f_D(S)$ is a quartic surface in $\mathbb{P}^3$. By Corollary 2.4 the Galois group must be cyclic, which is a contradiction. □

Claim 4.17. *The case $(3, 2; 2, 4)$ does not occur.*

Proof. Suppose otherwise. Then, there exists a smooth surface $S_2 = S/G_2$, which is a double covering of $S/G \cong \mathbb{P}^2$ branched along Δ_1. However, $\deg \Delta_1$ is odd, hence the double covering cannot exist. This is a contradiction. □

Thus only the case $n_1 = 3$ and $n_2 = 2$ remains, which corresponds to $G \cong Z_3 \times Z_2 \cong Z_6$. Combining the results above, we complete the proof. The last assertion $S \cong S_{(222)}$ will be proved in Theorem 5.12 below.

5. Particulars

In this section we describe all the surfaces in Theorem 4.1. We study the Galois embeddings of S in detail for $D^2 = 2m$, where $m = 2, 3$ and 4. Let C be a general member of the complete linear system $|D|$. Then we have $g(C) = m + 1$, where $g = g(C)$ is the genus of C and $\dim \mathrm{H}^0(S, \mathcal{O}(D)) = g + 1$. So $f_D(S)$ is assumed to be embedded in $\mathbb{P}^g$.

CASE 1. $g = 3$

Assume $|G| = D^2 = 4$. Then $G \cong Z_4$. Taking suitable homogeneous coordinates such that the Galois point be $X_0 = X_1 = X_2 = 0$, then $\beta_1(\sigma)$, which is the projective transformation (3) defined in Section 2, can be expressed as a diagonal matrix $[1, 1, 1, e_4]$. Since the defining equation of $f_D(S)$ is invariant by this transformation, we infer readily the following.

Theorem 5.1. *We have that $g = 3$ if and only if $G \cong Z_4$. In this case the defining equation of $f_D(S)$ can be given by $X_3^4 + F_4(X_0, X_1, X_2) = 0$, where $F_4(X_0, X_1, X_2)$ is a form of degree four.*

Remark 5.2. The maximal number of Galois points for the surface in Theorem 5.1 is four. And it is four if and only if it is the Fermat quartic (Example 3.4).

We can show a relation between the possibility of Galois embedding and the Picard number $\rho(S)$ for S.

Lemma 5.3. *For the surface S in Theorem 5.1 we have $\rho(S) \geq 2$.*

Before proceeding with the proof we note the following.

Remark 5.4. A smooth quartic plane curve Δ has at least 16 bitangent lines.

Proof. Let $\hat{\Delta}$ be the dual curve of Δ. Then we have $\deg \hat{\Delta} = 12$ and the genus of smooth model of $\hat{\Delta}$ is 3. Let T_P be the tangent line to Δ at P. If the intersection number of T_P and Δ at P is $i + 2 \geq 3$, then P is said to be an i-flex. Letting a_i be the number of i-flexes of Δ $(i = 1, 2)$, we see that $\hat{\Delta}$ has a_1-pieces of $(2, 3)$ cusps and a_2-pieces of $(3, 4)$ cusps. Referring to [2, Theorem 6.11], we get $a_1 + 2a_2 = 24$. If b is the number of nodes of $\hat{\Delta}$, then, applying the genus formula [2, Theorem 9.1], we get $b \geq 16$. Since a node of $\hat{\Delta}$ corresponds to a bitangent line of Δ, the proof is complete. □

Let Δ be the branch locus of $\pi : S \longrightarrow \mathbb{P}^2$, which is a smooth quartic curve. Let ℓ be a bitangent line to Δ and we consider $\pi^*(\ell)$.

Claim 5.5. *The curve $\pi^*(\ell) = \Gamma$ is a sum of two (-2)-curves.*

Proof. Let P_1 and P_2 be $\pi^{-1}(\ell \cap \Delta)$. Suppose Γ is irreducible. Then, it is not difficult to see by local consideration that it has two singular points P_i $(i = 1, 2)$. Each P_i is locally isomorphic to the singularity defined by $y^2 = x^4$. Let $\mu : \widetilde{\Gamma} \longrightarrow \Gamma$ be the resolution of singularities. Then $\pi|_\Gamma \cdot \mu : \widetilde{\Gamma} \longrightarrow \ell$ is a cyclic Galois covering of degree 4. Then, by Riemann-Huwitz formula we have $2g(\widetilde{\Gamma}) - 2 = 4(-2) + 4 = -4$, which is a contradiction. Hence we have $\pi^*(\ell) = \Gamma_1 + \Gamma_2$, where Γ_i $(i = 1, 2)$ is a (-2)-curve and $\Gamma_1.\Gamma_2 = 4$. □

From this claim Lemma 5.3 is clear.

CASE 2. $g = 4$

In this case $|G| = 6$. So $G \cong Z_6$ and put $G = \langle \sigma \rangle$.

Theorem 5.6. *If $G \cong Z_6$, then $f_D(S)$ is a $(2, 3)$-complete intersection, furthermore the defining equation of $f_D(S)$ can be given by $F_2(X_0, X_1, X_2) + {X_3}^2 = F_3(X_0, X_1, X_2) + {X_4}^3 = 0$, where $F_i(X_0, X_1, X_2)$ is a form of X_0, X_1, X_2 with degree i $(i = 2, 3)$ such that each curve $F_i = 0$ in $\mathbb{P}^2$ has no singular points.*

Proof. Since the embedding is given by $|D|$, where $D^2 = 6$, the surface $f_D(S)$ is the smooth complete intersection. In the proof of Theorem 4.1, in case $|G| = 6$, we have shown that $d_1 = n_1 = 3$ and $d_2 = n_2 = 2$. So that σ^2 (resp. σ^3) is identity on C_1 (resp. C_2). We have two covering maps $f_i : S_i := S/\langle \sigma^i \rangle \longrightarrow \mathbb{P}^2$, where $i = 2$ and 3. The f_i is a Galois covering of degree i branched along Δ_i. Put

$g_i : S \longrightarrow S_i$. Then we have $f_2 g_2 = f_3 g_3$. Since Δ_2 and Δ_3 have normal crossings, the fiber product $S_2 \times_{\mathbb{P}^2} S_3$ is smooth. Since S is also given by the double covering of S_3 branched along $g_3(C_2)$, we see that S is isomorphic to the fiber product $S_2 \times_{\mathbb{P}^2} S_3$. Furthermore, by taking a suitable coordinates on $\mathbb{P}^2$, we can assume S_2 is defined by $X_3^2 + F_2(X_0, X_1, X_2) = 0$ and S_3 by $X_4^3 + F_3(X_0, X_1, X_2) = 0$. This proves the theorem. □

There is some relation between a Galois embedding and the trivilality of the symplectic group G_s. From the following Corollary 5.7 to Corollary 5.11
<u>we do not assume that G is abelian</u>.

Corollary 5.7. *Suppose S has a Galois embedding. Then, G_s is trivial if and only if the embedding is given by a divisor D such that $D^2 = 4$ or 6.*

Proof. If G_s is trivial, then G is cyclic, hence $G \cong Z_4$ or Z_6. Conversely, if $G \cong Z_4$ or Z_6, then the defining ideal of $f_D(S)$ and the generator σ are given in Theorems 5.1 or 5.6. Referring to [5, Lemma 2.1], we conclude $|\Gamma_m| = 4$ or 6, where Γ_m is the cyclic group in (6). Thus G_s is trivial. □

We consider the Picard number $\rho(S)$ for the surfaces S in Theorem 5.1 and 5.6.

Lemma 5.8. *If S is the surface in Theorem 5.6, then $\rho(S) \geq 2$.*

Proof. Let σ be a generator of G and consider $S/\langle \sigma^3 \rangle$, which is a rational surface containing (-1)-curve. In fact, it is a smooth cubic in $\mathbb{P}^3$. Then we infer readily that S has a (-2)-curve. □

Let T_X be the transcendental lattice for a $K3$ surface X. Machida and Oguiso [4] prove the following:

Lemma 5.9. *Let X be a $K3$ surface and G be a finite automorphism group of X. Assume that* rank $T_X \geq 14$. *Then $G_s = \{1\}$, or equivalently, $G \cong \Gamma_m$.*

As we expect, a "general $K3$ surface" does not have a Galois embedding. Indeed, combining the results above, we deduce the following assertion.

Theorem 5.10. *If $\rho(S) = 1$, then S has no Galois embeddings.*

Corollary 5.11. *If S has a Galois embedding and $\rho(S) \leq 8$, then it is isomorphic to $S_{(4)}$ or $S_{(23)}$. Hence $G \cong Z_4$ or Z_6.*

Then, what can we say about a Galois embedding when $\rho(S)$ is large ? Can we say that S has the Galois embedding in the case where $\rho(S)$ is the maximal possible 20 ?

<u>CASE 3. $g = 5$</u>
In this case $|G| = 8$. So $G \cong Z_2^3$.

Theorem 5.12. *If $G \cong Z_2^3$, then S is a double covering of $S_{(22)}$, where $S_{(22)}$ is a rational surface of $(2,2)$-complete intersection in $\mathbb{P}^4$. Furthermore we have the following sequence of surfaces:*

$$S \xrightarrow{\pi_1} S_{(22)} \xrightarrow{\pi_2} S_{(2)} \xrightarrow{\pi_3} \mathbb{P}^2, \tag{6}$$

which have the following properties.

(1) *π_i ($i = 1, 2, 3$) is a double covering and $\pi = \pi_3 \cdot \pi_2 \cdot \pi_1$.*
(2) *$S_{(22)}$ is a surface of $(2, 2)$-complete intersection of $\mathbb{P}^4$.*
(3) *$S_{(2)}$ is a smooth conic in $\mathbb{P}^3$.*

Further more, the defining equation of $f_D(S)$ can be given by $X_3^2 + F_{23}(X_0, X_1, X_2) = X_4^2 + F_{24}(X_0, X_1, X_2) = X_5^2 + F_{25}(X_0, X_1, X_2) = 0$, where $F_{2i}(X_0, X_1, X_2)$ is a form of X_0, X_1, X_2 with degree 2, such that each curve $F_{2i} = 0$ ($i = 3, 4, 5$) in $\mathbb{P}^2$ has no singular points. In particular S is isomorphic to $S_{(222)}$.

Proof. The proof is done by the same way as the one of Theorem 5.6. In this case we have $d_i = n_i = 2$ ($i = 1, 2, 3$). Let $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and $G_i = \langle \sigma_i \rangle$. Put $S_i = S/\langle \sigma_i \rangle$ and $S_{ij} = S/\langle \sigma_i, \sigma_j \rangle$, where $i \neq j$. Then S is a double covering of S_i and so is S_i of S_{ij}, and S_{ij} is a double covering of $\mathbb{P}^2$ branched along Δ_k, where i, j, k are mutually distinct. It is easy to see that S_i is isomorphic to the fiber product $S_{ij} \times_{\mathbb{P}^2} S_{ik}$ and hence S is isomorphic to $(S_{12} \times_{S_1} S_{13}) \times_{\mathbb{P}^2} S_{23}$. In particular S is isomorphic to $S_{(222)}$. □

Remark 5.13. In the case where G is not abelian and $g = 4$ or $g = 5$, we can show that G is isomorphic to the dihedral group. Furthermore such a $K3$ is obtained as a Galois closure surface of some rational surface. The research for non-abelian case will be done in the forthcoming paper.

There are a lot of problems concerning our theme, we pick up some of them.

Problems.

(1) How many Galois subspaces do there exist for one Galois embedding and how is their arrangement? In the case of a smooth quartic surface in $\mathbb{P}^3$, see Remark 5.2. Then, how is the case for $(2, 3)$-complete intersection or $(2, 2, 2)$-complete intersection?
(2) Does there exist a $K3$ surface S on which there exist two divisors D_i ($i = 1, 2$) such that they give Galois embeddings and $D_1^2 \neq D_2^2$?
(3) Does each singular $K3$ surface have a Galois embedding ?

Acknowledgement

The author expresses his gratitude to Hiroyasu Tsuchihashi, who gave him useful information.

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